Question 1. Let $\mathbb{R}^\omega$ be the countably infinite product of $\mathbb{R}$ with itself, and let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of sequences which are eventually zero, that is, $(a_i)_{i=1}^\infty$ such that only finitely many a_i's are nonzero. Determine the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ with respect to the box topology, the uniform topology, and the product topology on $\mathbb{R}^\omega$.

Answer.

(*i*) The product topology: A basic open set of $\mathbb{R}^\omega$ is of the form

$$U = U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots$$

where U_i are open sets in $\mathbb{R}$. Now take any $x = (x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots) \in \mathbb{R}^\omega$, and any basic open set U as above containing x. Let $y = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$. Now, it is easy to see that $y \in U$. Also, $y \in \mathbb{R}^\infty$. Hence $\mathbb{R}^\infty$ is dense in $\mathbb{R}^\omega$ in the product topology.

(*ii*) The box topology : Every basic open set is of the form

$$W = W_1 \times W_2 \times \cdots \times W_i \times W_{i+1} \times \cdots$$

Now take any $x = (x_1, x_2, \ldots, x_i, x_{i+1}, x_{i+2}, \ldots) \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, then $x_n \neq 0$ for infinitely many n. In particular, if

$$W_n = \begin{cases} \mathbb{R} \setminus \{0\} & \text{if } x_n \neq 0 \\ (-1, 1) & \text{if } x_n = 0 \end{cases}$$

then if $W = \prod W_n$ as above, then $x \in W$ and

$$W \cap \mathbb{R}^\infty = \emptyset$$

Hence $\mathbb{R}^\infty$ is closed in the box topology.

(*iii*) The uniform topology: Consider $\mathbb{R}^\omega$ with the uniform topology and let d be the uniform metric. Let $C \subset \mathbb{R}^\omega$ be the set of sequences that converge to 0. Claim, $\overline{\mathbb{R}^\infty} = C$.

Let $(x_n) \in \mathbb{R}^\omega - C$ be a sequence that does not converge to 0. This means that there is some $1 > \varepsilon > 0$ such that $|x_n| > \varepsilon$ for infinitely many n. Then $B_d((x_n), \varepsilon/2) \subset \mathbb{R}^\omega - C$. Thus, $\mathbb{R}^\omega - C$ is open and hence C is closed. Therefore, $\overline{\mathbb{R}^\infty} \subset C$ as $\mathbb{R}^\infty \subset C$.

On the other hand, let $(x_n) \in C$. For any $1 > \varepsilon > 0$ we have $|x_n| < \varepsilon/2$ for all but finitely many n. Thus $B_d((x_n), \varepsilon) \cap \mathbb{R}^\infty \neq \emptyset$. Thus, $(x_n) \in \overline{\mathbb{R}^\infty}$ and hence $C \subset \overline{\mathbb{R}^\infty}$. This completes the prove.

Question 2. Consider $\mathbb{Z}$ as a normal subgroup of the additive group $\mathbb{R}$ of real numbers. Prove that the group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the group S^1 as topological groups.

Answer: Let $f : \mathbb{R} \to S^1$ defined by $f(t) = e^{i2\pi t} = \cos(2\pi t) + i \sin(2\pi t)$. Here $\mathbb{R}$ is a group with respect to addition and S^1 is a group with respect to complex multiplication. First we prove that f is a surjective group homomorphism. Clearly, f is surjective since for every $z \in \mathbb{C}$ on the unit circle, $z = e^{i2\pi t}$ for some $t \in \mathbb{R}$. Now, we will show that f is a group homomorphism.

$$f(t + s) = e^{2\pi i(t+s)} = e^{2\pi it+2\pi is} = e^{2\pi it} \cdot e^{2\pi is} = f(t) \cdot f(s).$$

So f is a group homomorphism. Observe Ker $f = \{t \in \mathbb{R} | e^{2\pi t} = \cos(2\pi t) + i \sin(2\pi t) = 1\} = \{t \in \mathbb{R} | \cos(2\pi t) = 1\} = Z$. It follows from the first isomorphism theorem that

$$\mathbb{R}/\text{Ker } f \cong \text{Im } f \Leftrightarrow \mathbb{R}/\mathbb{Z} \cong \text{Im } f = S^1.$$

Since $\mathbb{R}/\mathbb{Z}$ and S^1 are both topological groups, $\mathbb{R}/\mathbb{Z}$ is isomorphic to S^1 as topological group.

Question 3. Give an example (with details) of a connected topological space which has infinitely many path connected components. If X is a locally path connected topological space and U is a connected open subset of X, then prove that U is path connected.

Answer.

(i) The ordered square I_o^2 is a connected topological space but it has infinitely many path connected components. Being a linear continuum, the ordered square is connected.

Claim: $\{\{x\} \times I | x \in I\}$ be the path components of I_o^2.

For any two points $x \times a$ and $x \times b$ in $\{x\} \times I$ there is a path $f : [a, b] \to \{x\} \times I$ defined by $f(y) = x \times y$ from $x \times a$ to $x \times b$. Therefore, $\{x\} \times I$ is path connected.

Let $p = x \times 0$ and $q = y \times 1$ be two points in $\{x\} \times I$ and $\{y\} \times I$ respectively, where $x < y$. if possible, let $f : [a, b] \to [x, y] \times I$ is a path joining $p = x \times 0$ and $q = y \times 1$. Then, the image set $f([a, b])$ must contains every point in $[x, y] \times I$, by the intermediate value theorem. Therefore, for each $r \in [x, y]$ the set $U_r = f^{-1}(r \times (0, 1))$ is a non empty open subset of $[a, b]$. Choose, for each $r \in [x, y]$, a rational number $q_r \in U_r$. Since the sets U_r are disjoint, the map $r \to q_r$ is an injective map from the closed interval $[x, y]$ into the set of rational number $\mathbb{Q}$. This contradict the fact that the closed interval $[x, y]$ is uncountable.

(ii) Let $x \in U$ and let $V \cap U$ be an open set in U containing x for some open set V in X. Since U is open in X, $V \cap U$ is open in X. Therefore, there is a path connected neighborhood of x contained in $V \cap U$. Therefore, U is a locally path connected space. By Theorem 25.5, in a locally path connected space, the components and the path components are the same. Since U is connected, U is the only component and hence U is the only path component of U. Therefore, U is path connected.

Question 4. Let I_o^2 be the oredered square. Is I_o^2 locally path connected? locally connected? compact?

Answer.

(i) The ordered square is locally connected: just observe that any neighborhood U of any point $x \times y$ contains an interval of the form $(a \times b, c \times d)$ for $a \times b < x \times y < c \times d$ by definition of the order topology. By Theorem 24.1 in the book, an (open) interval of a linear continuum is connected, so $(a \times b, c \times d)$ is connected. Hence the ordered square is locally connected.

(ii) The ordered square is not locally path-connected: consider any point of the form $x \times 0$. By definition of the order topology, any open neighborhood of $x \times 0$ must be of the form $U = (a \times b, c \times d)$ where $a \times b < x \times 0$. Since the second coordinate of $x \times 0$ is 0 this means that $a < x$. In particular U contains all points of the form $y \times z$ for $a < y < x$. Fix one such y_0 and choose any $z_0 \in [0, 1]$. We claim that there is no path which connects $y_0 \times z_0$ with $x \times 0$. Suppose there is such a path; then by the intermediate value theorem, all points in $I \times I$ between $y_0 \times z_0$ and $x \times 0$ are in the image of the path. However there are uncountably many elements in the interval (y_0, x), so the same argument as in the answer of question 3(i), shows that this is impossible. Hence the ordered square is not locally path-connected.

(iii) The ordered square I_o^2 is compact: Let $\mathcal{U}$ be an open cover of I_o^2. Choose a point $x \in I$. Then $\mathcal{U}$ is also an open cover of $\{x\} \times I$. Since $\{x\} \times I \cong I$ and I is compact, this open cover has a finite subcover $\mathcal{U}_x$. Let U_x be the union of the elements of the finite subcover. If $x \in (0, 1)$, then there exist p_x, q_x such that $0 \leq p_x < x < q_x \leq 1$ and $p_x \times 0, q_x \times 0 \in U_x$. If $x = 0$ then there is such a q_x and if $x = 1$ the there is such a p_x. Let

$$V_x = \begin{cases} [0, q_x) & \text{if } x = 0 \\ (p_x, q_x) & \text{if } x \in (0, 1) \\ (p_x, 1] & \text{if } x = 1 \end{cases}$$

Then $\{V_x | x \in I\}$ is an open cover of I. Since I is compact, it has a finite subcover, say, $\{V_{x_1}, V_{x_2}, \ldots, V_{x_n}\}$. Then $\mathcal{U}_0 \cup \mathcal{U}_{x_1} \cup \mathcal{U}_{x_2} \cdots \cup \mathcal{U}_{x_n} \cup \mathcal{U}_1$ is a finite subcover of $\mathcal{U}$ which covers the ordered square.

Question 5. State and prove the Lebesgue number lemma.

Statement: If the metric space (X, d) is compact and an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ; is contained in some member of the cover.

Proof: Let $\mathcal{U}$ be an open cover of X. Since X is compact we can extract a finite subcover $\{A_1, \ldots, A_n\} \subseteq \mathcal{U}$.

For each $i \in \{1, \ldots, n\}$, let $C_i := X \setminus A_i$ and define a function $f : X \to \mathbb{R}$ by $f(x) := \frac{1}{n}\sum_{i=1}^{n} d(x, C_i)$.

Since f is continuous on a compact set, it attains a minimum δ. The key observation is that $\delta > 0$. If Y is a subset of X of diameter less than δ, then there exist $x_0 \in X$ such that $Y \subseteq B_\delta(x_0)$, where $B_\delta(x_0)$ denotes the ball of radius δ centered at x_0 (namely, one can choose as x_0 any point in Y). Since $f(x_0) \geq \delta$ there must exist at least one i such that $d(x_0, C_i) \geq \delta$. But this means that $B_\delta(x_0) \subseteq A_i$ and so, in particular, $Y \subseteq A_i$.

(Reference: `https://en.wikipedia.org/wiki/Lebesgue%27s_number_lemma`)